

Spurious fixed points in frustrated magnets

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We analyze the validity of perturbative estimations obtained at fixed dimensions in the study of frustrated magnets. To this end we consider the five-loop β -functions obtained within the minimal subtraction scheme and exploited without ϵ -expansion both for frustrated magnets and for the well-controlled ferromagnetic systems with a cubic anisotropy. Comparing the two cases it appears that the fixed point supposed to control the second order phase transition of frustrated magnets is very likely an unphysical one. This is supported by the non-Gaussian character of this fixed point at the upper critical dimension $d = 4$. Our work confirms the weak first order nature of the phase transition and constitutes a step towards a unified picture of existing theoretical approaches to frustrated magnets.

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Introduction. Although undoubtedly successful to describe the critical behavior of $O(N)$ -like models, perturbative field theory is still unable to provide a clear, non-controversial understanding of the physics of more complex models among which are the famous Heisenberg or XY frustrated magnets (see [1] and references therein). At the core of the problem is that different kinds of perturbative approaches, performed up to five- or six-loop order, lead to contradictory results: first order phase transitions are predicted within the ϵ (or pseudo- ϵ)-expansion [2, 3, 4] whereas a second order transition is found in the fixed-dimension (FD) perturbative approaches performed either in the minimal-subtraction ($\overline{\text{MS}}$) scheme *without* ϵ -expansion [5] or in the massive scheme [6]. In fact FD results for frustrated magnets are neither supported by experiments nor by Monte Carlo simulations [7, 8, 9, 10] (see however [5] where a scaling behavior is found). They also disagree with the predictions obtained from the non-perturbative renormalization group (NPRG) approach [1, 11] that predicts (weak) first order phase transitions in $d = 3$ in agreement with the ϵ -expansion analysis.

In this letter we shed light on the discrepancies encountered in perturbative approaches to frustrated magnets by showing that the FD approaches very likely lead to incorrect predictions as for the critical physics in three dimensions. Our key-point is easy to grasp already for the simplest — $O(N)$ — model. In this case, the (non-resummed) renormalization group (RG) β -function at L loops is a polynomial in its coupling constant u of order $L + 1$. Thus, it admits $L + 1$ roots u^* , $\beta(u^*) = 0$, that are either real or complex. Within the ϵ -expansion approach, the only non-trivial fixed point (FP) retained is such that $u^* \sim \epsilon$ where $\epsilon = 4 - d$. On the contrary, in the FD approaches, no real root can be *a priori* discarded. As a result, the generic situation is that the number of FPs as well as their stability vary with the order L : at a

given order, there can exist several real and stable FPs or none instead of a single one. This artefact of the FD approach is already known and was first noticed in the massive scheme in $d = 3$ [12]. The way to cope with it is also known: resumming the perturbative expansion of $\beta(u)$ (see e.g. [13, 14]) is supposed both to restore the nontrivial Wilson-Fisher FP and to suppress the non-physical or “spurious” roots. This is indeed what occurs for the $O(N)$ model for which the FP analysis performed on the resummed β -function of FD approaches enables to discriminate between physical and “spurious” FPs. We argue that the situation is very different for frustrated magnets. Indeed, considering the β -functions derived at five loops in the $\overline{\text{MS}}$ scheme and using the same resummation procedure as Calabrese *et al.* [5] we show that the FP found in $d = 3$ without expanding in ϵ is spurious although it persists after resummation. Our conclusion is based on two main facts: (i) analyzed with the same FD approach a similar FP is found for the ferromagnetic system with cubic anisotropy in contradiction with its well established critical physics (ii) there are strong indications that these FPs survive in the upper critical dimension $d = 4$ where they are found to be non-Gaussian which raises serious doubts on their actual existence.

Resummation. To investigate the five-loop β functions derived in the $\overline{\text{MS}}$ scheme we resum them, following [5], using a conformal mapping Borel transform suitable for series involving two coupling constants u_1 and u_2 . We consider any function f of u_1 and u_2 as a function of u_1 and $z = u_2/u_1$, supposed to be known through its series expansion:

$$f(u_1, z) = \sum_n f_n(z) u_1^n. \quad (1)$$

An important hypothesis underlying the procedure of resummation used here and in [5] is that one can safely resum (1) in the u_1 direction while keeping z fixed, *i.e.*

without resumming with respect to u_2 . Under this hypothesis a resummed expression associated with f reads:

$$f_R(u_1, z) = \sum_n d_n(\alpha, a(z), b; z) \int_0^\infty dt \frac{e^{-t} t^b [\omega(u_1 t; z)]^n}{[1 - \omega(u_1 t; z)]^\alpha} \quad (2)$$

with $\omega(u; z) = (\sqrt{1 + a(z)u} - 1)/(\sqrt{1 + a(z)u} + 1)$ and where the coefficients $d_n(\alpha, a(z), b; z)$ are computed so that the re-expansion of the r.h.s. of (2) in powers of u_1 coincides with that of (1). In principle the parameters $a(z)$, b and α are determined (i) by the asymptotic behavior of f_n : $f_{n \rightarrow \infty} \sim (-a(z))^n n! n^b$ and (ii) by the strong coupling behavior of f : $f(u_1 \rightarrow \infty, z) \sim u_1^{\alpha/2}$. However in general only a is known and the other parameters are considered as free or variational.

Frustrated magnets. The Hamiltonian relevant for frustrated systems is given by:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} [(\partial \phi_1)^2 + (\partial \phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] + \frac{u_1}{4!} [\phi_1^2 + \phi_2^2]^2 + \frac{u_2}{12} [(\phi_1 \cdot \phi_2)^2 - \phi_1^2 \phi_2^2] \right\} \quad (3)$$

where the ϕ_i are N -component vector fields. We apply the resummation procedure described above without ϵ -expansion [15] to the β_{u_i} functions, $i = 1, 2$, obtained at five loops in the $\overline{\text{MS}}$ scheme [5]. More precisely, as in [5], we resum $(\beta_{u_i}(u_1, z) + \varepsilon u_i)/u_1^2$, $i = 1, 2$, instead of $\beta_{u_i}(u_1, z)$ [26]. In this model the region of Borel-summability is given by $u_1 - u_2/2 > 0$ while $a(z) = 1/2$ and b and α are typically varied in the ranges [6, 16] and $[-0.5, 2]$. As in all other approaches, one finds, in agreement with [5], that there exists a curve (parametrized by $N_c(d)$ or its reciprocal $d_c(N)$) such that for $d < d_c(N)$ a stable FP C_+ governs the critical properties of the system. The curves $N_c(d)$ obtained within this scheme, the NPRG approach and the ϵ -expansion are given in Fig.1. This figure sums up the discrepancies between these approaches. The S-like shape of the curve $N_c^{\text{FD}}(d)$ obtained within the perturbative FD approach is such that C_+^{FD} exists for all $N \geq 2$ in $d = 3$ contrary to the other approaches in which a FP C_+ exists only for $N > N_c(d = 3) \simeq 5$ [1].

The main question is to know whether one can trust the results obtained at FD, in particular the existence of FPs in $d = 3$ for $N = 2, 3$. To answer this question we examine the convergence of the FD perturbative results through their sensitivity with respect to variations of the resummation parameters b and α as well as the order L of the computation. We concentrate our study on the (real part of the) stability critical exponent ω at the FP C^+ which is focus [16]. We have checked that similar behaviors are obtained for the exponent ν [17]. In practice, following [18], we optimize $\omega(\alpha, b, L)$ by choosing α such that $\omega(L + 1) - \omega(L)$ is minimal and b such that

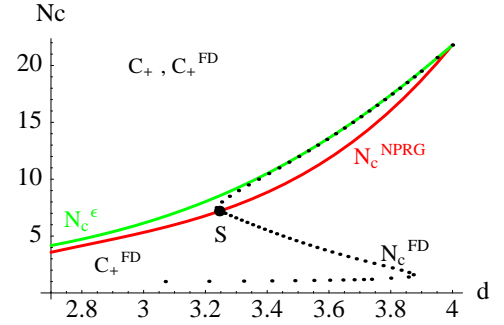


FIG. 1: Curves $N_c(d)$ obtained within the ϵ -expansion (N_c^ϵ), the $\overline{\text{MS}}$ scheme without ϵ -expansion (N_c^{FD}) and the NPRG approach (N_c^{NPRG}). The resummation parameters for the $\overline{\text{MS}}$ curve are $a = 1/2$, $b = 10$ and $\alpha = 1$. The part of the curve N_c^{FD} below S corresponds to a regime of non-Borel-summability.

$\omega(b)$ is (quasi-)stationary. In Fig.2 we display $\omega(b)$ in the Heisenberg case for $L = 4$ and 5 for typical values of α . At a given order the variations of ω with b are small around the stationary point and do not exceed 30% from four to five loops in agreement with [5]. Similar results are obtained in the XY case. From this analysis alone, one could conclude that a FP indeed exists in $d = 3$ although with error bars on the critical exponents far larger than those obtained for the Ising model with the same methodology (much less than 1% at this order). Let us now study the cubic model along the same lines.

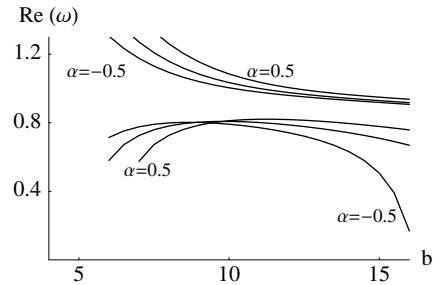


FIG. 2: The (real part of the) critical exponent ω as a function of b at five (upper curves) and four (lower curves) loops for $\alpha = -0.5, 0$ and 0.5 for the frustrated model ($N = 3$).

The cubic model. We now consider the ferromagnetic model with cubic anisotropy whose Hamiltonian is:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} [(\partial \phi)^2 + m^2 \phi^2] + \frac{u}{4!} [\phi^2]^2 + \frac{v}{4!} \sum_{i=1}^N \phi_i^4 \right\} \quad (4)$$

with ϕ a N -component vector field. The Hamiltonian (4) is used to study the critical behaviors of numerous magnetic and ferroelectric systems with appropriate order parameter symmetry (see e.g. [14, 19]). The β -functions are known at five-loop order in the $\overline{\text{MS}}$ scheme [20] and at six-loop order in the massive scheme [21]. Apart from

the Gaussian and an Ising FP, there exist two FPs: the $O(N)$ symmetric FP ($u^* \neq 0, v^* = 0$) and the mixed one M ($u^* \neq 0, v^* \neq 0$). The Ising and Gaussian FPs are both unstable for all values of N . The $O(N)$ FP is stable and M is unstable with $v^* < 0$ for $N < \tilde{N}_c$ and the opposite for $N > \tilde{N}_c$. \tilde{N}_c has been found to be slightly less than 3 [19, 21].

Let us now analyze the FP structure of the model (4) within the $\overline{\text{MS}}$ scheme without ϵ -expansion by applying the conformal mapping Borel transform (2). The parameter $a(z = v/u)$ entering in (2) is now given by $a(z) = 1 + z$ for $z > 0$ and $a(z) = 1 + z/N$ for $z < 0$ while the region of Borel-summability is given by the condition $u + v > 0$ and $Nu + v > 0$. Within this scheme one surprisingly observes that, in addition to the above mentioned usual FPs, there exist, in a whole domain of parameters b and α , several other FPs that have no counterpart in the ϵ -expansion. In particular, one of them that we call P (which is stable and such that $u^* > 0, v^* < 0$) exists for any value of $N \lesssim 7.5$ and lies in the region of Borel-summability $u + v > 0$. The presence of this FP, if taken seriously, would have important physical consequences since it would correspond to a second order phase transition with a new universality class. However as far as we know, no such transition has ever been reported and the existence of P has to be considered as an artefact of the FD analysis. Note finally that P is found to be a *focus* FP, a striking similarity with the frustrated case.

At this stage, it is very instructive to perform for P the same convergence analysis as we have done for C_+ in the FD analysis. In Fig.3 we give $\omega(b)$ for $L = 4, 5$ and for three different values of α . Surprisingly, we find results that display similar convergence properties as in the frustrated case (with a spreading of the results of order 40%).

This study shows that contrary to what one could naively deduce, the convergence properties displayed in Fig.3 do not characterize P as a genuine FP. Comparing the studies of the frustrated and cubic cases one is led to the same conclusion for C_+^{FD} for $N = 2, 3$ in $d = 3$. Thus, in order to conclude whether C_+^{FD} is a genuine FP or an artefact of the FD analysis, one has to resort to another criterion.

Fixed point at the upper critical dimension. To do this let us perform the following observation: the choice of ϕ^4 -like Hamiltonian in Eq.(3) relies on the physical hypothesis that the upper critical dimension is four. From the perturbative point of view, this means in particular that the theory is infrared trivial — controlled by the Gaussian FP — in four dimensions. Thus, by consistency, one must retain among all solutions of the FP equations in $d = 4 - \epsilon$ only those that are at a distance of order ϵ of the Gaussian FP. Therefore a practical way to check whether a FP found at a given dimension, $d = 3$

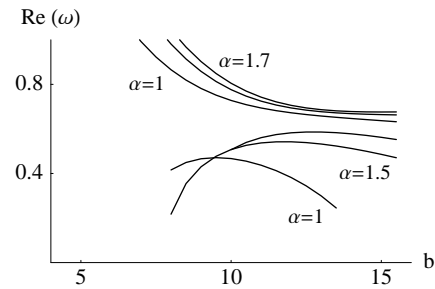


FIG. 3: The (real part of the) critical exponent ω as a function of b at five (upper curves) and four (lower curves) loops for $\alpha = 1, 1.5$ and 1.7 for the cubic model ($N=2$).

for instance, is a genuine FP or is just an artefact of perturbation theory is to follow it by continuity up to the upper critical dimension [3, 22]. If the FP survives as a non-Gaussian FP at this dimension this is a signal that it is spurious.

Let us apply this criterion simultaneously to P and C_+^{FD} . We present our results in Fig.4 where we have displayed the coordinates u^* and u_1^* associated to these FPs as functions of d . Manifestly, they both survive everywhere above $d = 3$ and are *not* Gaussian in $d = 4$. In the cubic case this is true for all values of N for which P exists while, in the frustrated case, this happens for all values of $N \lesssim 6$. Thus, according to our criterion, P is, as expected, always found to be spurious. As for C_+^{FD} for $N \lesssim 6$ and, in particular, for $N = 2, 3$ our criterion strongly suggests that it is also spurious in $d = 3$ [27]. Note that the FPs P and C_+^{FD} lie out of the region of Borel-summability in $d = 4$. Thus their coordinates cannot be determined accurately although their existence as non-Gaussian FPs is doubtless.

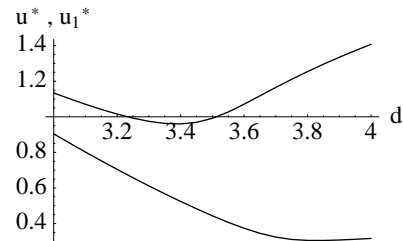


FIG. 4: The u^* coordinate of the FP P ($N = 2$, upper curve) and the u_1^* coordinate of the FP C_+^{FD} ($N = 3$, lower curve) as functions of d .

Conclusion. It appears from our study that the FP C_+^{FD} identified in the FD approach is very likely spurious. The transition should thus be of — possibly weak — first order in agreement with NPRG and ϵ -expansion approaches. It remains to explain the failure in the resummation procedure used in the FD approach, Eq.(2). As already emphasized this procedure relies on the hypothesis that resumming with respect to u_1 , keeping a poly-

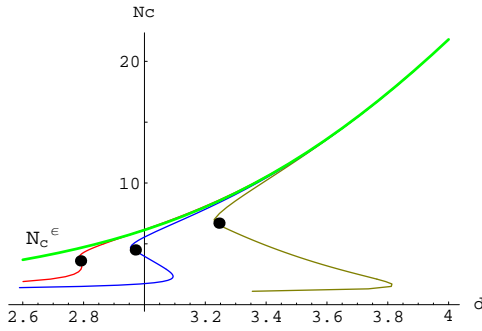


FIG. 5: Four curves $N_c^{\text{FD}}(d)$ for different values of the parameter a and the curve $N_c^\epsilon(d)$. From right to left $a = 0.5, 1.3, 1.5$. The parameters b and α are $b = 10$ and $\alpha = 1$. The parts of the curves below the black dots correspond to a regime of non-Borel summability.

nomial structure in u_2 , is sufficient. Alternatively a resummation of the series with respect to the two coupling constants could be required to obtain reliable results (see for instance [23] for the randomly diluted Ising model). Postponing these considerations for a future publication [17] we assume that the use of Eq.(2) as such is justified. Then a possible origin of the failure in the resummation procedure could be that the series considered are not known at large enough order to reach the asymptotic behavior. In this case there would be no reason to fix the parameter a at its asymptotic $1/2$ value and one should have to vary it as b and α to optimize the results [18]. We display in Fig.5 the curves $N_c^{\text{FD}}(d)$ for different values of a . The part corresponding to large values of N_c , typically for $N_c \gtrsim 6$, is almost insensitive to the variations of a whereas this is clearly not the case for smaller values of N_c . In particular, for sufficiently large values of a , typically $a \geq 1.5$, the S-like part is pushed below $d = 3$ so that $N_c^\epsilon(d)$, $N_c^{\text{FD}}(d)$ and $N_c^{\text{NPRG}}(d)$ are then compatible everywhere for $3 < d < 4$. Thus, under our hypothesis, all qualitative differences between the different approaches disappear [28] so that the problem would boil down to a question of order of computation. It is clear that our heuristic argument, although appealing, requires more rigorous developments [17].

Note finally that our present considerations surely pertains to the case of frustrated magnets in $d = 2$ [16]. Indeed we have checked that the FP found in $d = 2$ is continuously related to the FP C_+^{FD} in $d = 3$ which makes its existence doubtful. Our conclusions could also apply in other situations where FPs that have no counterpart in the ϵ -expansion is found, as it is the case, for instance, in QCD at finite temperature [24].

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- [26] Resumming the functions β_{u_i} leads to similar results.
- [27] Let us clarify a point specific to the frustrated systems. The curve $N_c^{\text{FD}}(d)$ can be separated into two parts by the point S , see Fig.1. If one follows C_+^{FD} along a path starting in $d = 3$, going to $d = 4$ and crossing $N_c^{\text{FD}}(d)$ above S , its coordinates become complex in $d = d_c(N)$ and go to zero for $d = 4$ where it is thus the Gaussian FP. If, on the contrary, the path crosses $N_c^{\text{FD}}(d)$ below S , C_+^{FD} changes from a stable focus to an unstable one at $d = d_c(N)$ and does not go to the Gaussian in $d = 4$. Thus, S is a singularity of $u_1^*(N, d)$ and $u_2^*(N, d)$. The FD results are therefore doubtful below S but trustworthy above where they are indeed found to be consistent with

the other approaches, see Fig.1.

- [28] One also observes that the shape of the curve $N_c^{\text{FD}}(d)$ for $a = 1.3$ is compatible with the results obtained in the massive scheme in $d = 3$ in which one finds fixed points

for all values one N but in the range $5.7(3) < N < 6.4(4)$ [25].